PERGAMON

International Journal of Solids and Structures 37 (2000) 81–92

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Plasticity, limit analysis and structural design

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Abstract

It is important for any discipline from time to time to take stock of its past achievements and to help put the current progress and future directions into perspective. In the case of research and education in plasticity and its impact on structural engineering, these assessments will help shape the directions of our future focus on research opportunities for the next decade. This is described in the present paper. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

What measuring stick should be used to assess the accomplishments of plasticity research for structural engineering profession in the last forty years? Should it be the volume of papers presented, or the number of journal articles published, or the number of Ph.D. theses produced or the number of new courses in the universities offered? I believe that the 'bottom line' for plasticity research should be the amount of research which finds its way into engineering practice. In the following, I shall first summarize briefly in tabular form the 'major advances' of structural engineering that can be attributed to the 'breakthrough' of solid mechanics in general, and plasticity in particular, in the last 40 years. These '*success stories*' fall into a number of broad categories. Within each category are several specific examples where new knowledge has been implemented in structural engineering and, in some measure, the structural engineering practice has been fundamentally changed. I will present some of these success stories from my own experience in the later part of this paper. Since much of my own research over the past 35 years has involved the interaction of *Mechanics*, *Materials* and *Computing*, my view on these developments is therefore strongly influenced by this background. I believe the major advances in solid mechanics and structural engineering in the last forty years are closely related to the interaction of these three areas as:

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PII: S0020-7683(99)00079-7

Solid Mechanics and Structural Engineering: (*Mechanics, Materials, Computing*)

- Elasticity
- Plasticity
- Finite Element
- Fracture Mechanics
- Reliability
- Composite Materials
- Integrated Lifecycle Simulation (*Modeling and Simulation*)

More detailed discussion on this subject can be found in the state-of-the-art paper by Chen (1998).

2. A brief historical sketch on elasticity and plasticity in structural design

2.1. Elasticity and slide-rule computing in the 1950's

This was the period of slide-rule computing in engineering practice, and structural design was based on elastic theory which assumes that structures display a linear response throughout their loading history, ignoring the post-yielding stage of behavior. In this design practice, the elastic analysis is used to compute forces and moments in the members of a structural system, while empirical mumbo-jumbo expressions based on full-scale tests of structural members or elements are used to proportion cross-sections of moment and axial load of members. The most important advances in structural analysis in this decade may be represented by the works of Hardy Cross and N. Newmark of the University of Illinois, and of S.P. Timoshenko of Stanford University. The *Moment Distribution Method*, developed by Hardy Cross in the 1930's, was a very efficient and practical method for the design of high-rise buildings during that period. The subsequent developments of *Numerical Procedures* by N. Newmark in the 1940's and the *Approximate Series* solutions by Timoshenko in the 1950's for structural members, frames, plates and shells form the foundations of modern theory of structural analysis and design.

2.2. Plasticity and computers in their infancy in the 1960's

In this period, the computer was in its infancy while the development of the classical theory of plasticity under the leadership of William Prager of Brown University was in its golden time. We had a rigorous theory but few practical solutions. The problem of determining the stresses in an elastic–plastic structural system requires the prior knowledge of the existing permanent deformations at that particular instant. Since, these, in turn, depend on the previous load history, in practice, most of the loading paths on a structure under consideration are somewhat arbitrary in nature; the question of obtaining the state of stress of a structure under a particular loading path is not only very difficulty but also rather meaningless. For engineering practice, we must develop a simple theory that is realistic and practical. There is nothing more practical than the simple *limit analysis* theorems and techniques developed by Drucker et al. of Brown University in the 1950's (Drucker and Prager, 1952). As a result, the *Plastic Design Methods* for steel structures were developed, refined and implemented into the AISC specifications in the 1960's, under the leadership of Lynn Beedle of Lehigh University in the United States, among many others.

The subsequent extension and expansion of the limit analysis based design applied to reinforced concrete structures as well as to stability problems in geotechnical engineering problems are far-reaching and most impressive. Some of these developments, as well as the directions of current efforts and future trends, will be described in the forthcoming.

3. A brief historical sketch on limit analysis in civil engineering

In the early 1960's, the computer was in its infancy while the theory of plasticity was in its golden time. As a result, the limit analysis methods were developed and widely applied to steel structures. This was the time I began my teaching career at Lehigh University almost 35 years ago. At Lehigh, I participated in full scale testing of steel structures for the practical development of *Plastic Design Methods in Steels*. The plastic design method was adopted officially by the American Institute of Steel Construction as the new design code in 1963. Only recently has the lower bound stress field method been adopted by the ACI specification for the design of structural members and joints in reinforced concrete, known as the truss model. These developments are briefly summarized in the following:

3.1. Limit analysis applied to steel structures

Building Frames — Plastic Design (1960's)

- 1963 AISC Plastic Design Specification was adopted.
- First-Order plastic-hinge analysis was used for frame design.
- Auto Stress Design was adopted in the 1980's for plastic design of bridges.

3.2. Limit analysis applied to reinforced concrete structures

Concrete is a very old construction material in civil engineering but we were not able to write down its stress–strain relation or the so-called constitutive relations under various combined stress and environmental conditions. This relationship for characterizing concrete's material properties must first be developed before any finite element analysis can be carried out for its structural computer simulation. As a result, the current design practice for reinforced concrete structures is a curious blend of elastic analysis to compute internal forces and moments in a structural system, then plasticity theory is used to size up the members with empirical expressions for member strength based on full scale tests. As pointed out in an article in *Concrete International* by J.G. MacGregor (1984):

> One of the most important advances in reinforced concrete design in the next decade will be the extension of plasticity based design procedures to shear, torsion, bearing stresses, and the design of structural discontinuities such as joints and corners. These will have the advantage of allowing a designer to follow the forces through a structure.

As illustrated in a recent book by Muttoni et al. (1997), the limit theory of perfect plasticity provides a consistent scientific basis, from which simple and, above all, clear models may be derived to determine the statical strength of reinforced concrete structures.

A.A. Gvozdev (1960) appears to be the first to develop the modern concept of limit analysis as it applied to reinforced concrete structures; while K.W. Johansen (1930) used the upper bound techniques of limit analysis, and developed the *Yield-Line Theory for Slab Design* for engineering practice. The application of stress fields to reinforced concrete beam design, based on the concept of lower bound theorem of limit analysis, was first proposed by Drucker (1961). This approach is now being extended and expanded to the design of concrete structures by Muttoni et al. (1997) in a very practical manner. Similarly, upper and lower bound concepts of limit analysis were used by J. Heyman (1966) to explain the modern view of the traditional design and construction of stone skeletons. A comprehensive summary on the applications of limit analysis techniques to reinforced concrete structures is given in a recent book by Nielsen (1998). A general overview of the developments of theory and applications of concrete plasticity can be found in the book by Chen (1982).

3.3. Limit analysis in soil mechanics

The solutions of classical stability problems in soil mechanics have been generally obtained by the well-known *limit equilibrium methods* of Terzaghi (1943) using Coulomb failure criterion. Similar results can be obtained rigorously through the applications of limit analysis of soil plasticity. The limit analysis results were summarized in the books by Chen (1975) and by Chen and Liu (1990). Modern developments of soil plasticity were centered at Cambridge University in the 1960's under the leadership of Professor Roscoe. His research team firmly established the *Critical State Soil Mechanics* (Schofield and Wroth, 1968). Many useful and practical solutions based on the critical state formulation have been obtained by the powerful computer-based finite element method. These results can be found in a recent book by Chen and Mizuno (1990). A brief discussion of the impact of the finite element method and theory of plasticity on structural engineering practice will be given in the forthcoming.

4. A brief historical sketch on the finite element method in structural engineering

In the 1970's, our computing power changed drastically with mainframe computing. The *Finite Element Method*s were well developed and widely used in structural engineering. We were able to apply the theory of stability and the theory of plasticity to simulate the actual behavior of structural members and frames with great confidence. It was the first time we were able to replace the costly full-scale tests with computer simulation. As a result, the limit state approach to design was advanced and new specifications were issues.

In the subsequent years, we had an energy crisis and offshore structural engineering and technology development became the central focus. In the meantime, our computing environment changed drastically and the cost of computing becomes almost insignificant as we entered into the PC and workstation era. This was also the period that we were able to solve almost any kind of structural engineering problems with computer simulation. But now, for the first time, the physical theory is lagging behind the computing power. We need to develop a more refined theory of constitutive equations for engineering materials for finite element types of applications. This marks the beginning of the modern development of *concrete plasticity* in the 1970's. The details of this development were described in the book by Chen (1982). A brief outline of this development will be described in the next section.

Three basic conditions for a valid solution (Ray Clough — U.C. Berkeley)

- Equilibrium Condition (Newton's Law or Physics)
  The *virtual work equation* is used exclusively to establish the relationship between the stress in an element to the *generalized stresses* at nodal points.
- Kinematic Condition (Continuity or Logic)
  The *shape function* is introduced to establish the relationship between the strain in an element to the *generalized strains* at the nodal points.
- Constitutive Relations (Material or Experiment)
  The *theory of plasticity or viscosity* is used to relate the generalized stresses to generalized strains through the use of constitutive equations of engineering materials. The two-volume treatise by Chen (1994a, 1994b), covers most of these developments, among others (Desai and Siriwardane, 1983; Chen and Baladi, 1985).

The following is a brief summary in tabular form of the impacts of the applications of finite element methods with plasticity theory on structural engineering practice.

4.1. F.E. applied to steel structures

1970s: Development of Member Strength Equations

- Beam strength equation — Beam Design Curve
- Column strength equation — Column Design Curve
- Beam–Column strength equation — Beam–Column Interaction Design Curve
- Biaxially loaded column strength equation for plastic design in steel building frames

These developments were summarized in the two-volume treatise by Chen and Atsuta (1976, 1977).

1980s: Limit State Approach to Design

- Development of reliability-based codes.
- The publication of the 1986 AISC/LRFD Specification.
- The introduction of the second-order elastic analysis to the design codes.
- The explicit consideration of semi-rigid connections in frame design (PR Construction).

These developments were summarized in the book by Chen and Lui (1992).

1990s: Structural System Approach to Design

- Second-Order inelastic analysis for frame design was under intense development.
- The theory of stability is combined with the theory of plasticity for direct frame design.
- The advanced analysis considers explicitly the influence of structural joints in analysis/design process.

These developments were summarized in the book by Chen and Toma (1994).

4.2. F.E. method applied to offshore structures (an illustrative example)

4.2.1. Research behind the success of the offshore structures

The offshore concrete structures constructed in the 1970's for the North Sea oil development were analyzed extensively with the finite element method. Some of the highlights of the analysis process are summarized in the following:

- Solve 100,000 simultaneous equations.
- Designed for a 30 m wave with the platform located in a 300 or 1000 ft deep water.
- Consider 25,000 load combinations.
- Use supercomputer for computing.
- Assume the material to be linearly elastic.

- Cost \$7M to develop the computer program.
- Require 250 engineers to input the data.

4.2.2. Failure experience, the problem

The structure failed during the installation process. The reasons for the failure of the analysis are due to:

- Did not consider that the concrete will crack after overloading and redistribute the stresses.
- The anchorage of the reinforced bars was found in the tension zone after the crack of concrete.
- Costs of the failure exceed \$0.5B for the structure and \$1B for the overall economy.
- The computed shear force by F.E. is about 60% of simple beam hand calculations.

4.2.3. Lessons learned

The subsequent analysis and design for a successful construction of the platform consider the following improvements:

- Improve the modeling on strength and deformation of R.C. element under all possible load combinations and torsion.
- Carry out large-scale element tests for both strength and fatigue.
- Conduct biaxial compression/tension tests:
  - Strength increased by lateral compression, 30%.
  - Strength decreased by every cycle of tension.
- Put much more steels in the shells.
- Use concrete with slump of 260 mm (10 in) instead of 120 mm (4.7 in) to get through.
- Shells are too heavy for installation. Use light-weight concrete to reduce weight.

4.2.4. Conclusions remarks

- Engineers need to develop a good material model for a heavily reinforced concrete plate element.
- Need to do simple hand calculations to check the computer solutions.
- Need experienced engineers to do hand calculation check.
- Need to consider partial failure analysis, like cracks to see possible redistribution.

5. Concrete plasticity: recent developments

5.1. Failure criteria as a start

The early effort to develop a plasticity model for concrete materials has been centered in the search for a suitable failure surface. A failure criterion of Coulomb type with a tension cutoff has been used widely in engineering practice. Based on the knowledge concerning the failure surface, a variety of failure criteria have been proposed in the past 20 years. Most of these criteria are discussed in the book by Chen (1982), where they are classified by the number of material constants appearing in the expressions as one-parameter through five-parameter models. All include the strong influence of the normal stress on the shear required in the plane of sliding. All assume the isotropy of the material and convexity of the failure surface.

5.2. Work-hardening as a next step

Once a mathematically and physically attractive failure criterion has been established, the next step is to use the work-hardening theory to establish the stress–strain relation in the plastic range. The relatively sophisticated model developed by Han and Chen (1985) — the model of non-uniform hardening plasticity — illustrates this step. This non-uniform hardening plasticity model:

1. Adopts the most sophisticated five-parameter failure surface of Willam–Warnke as the bounding surface,
2. Assumes an initial yield surface with a shape that is different from the failure surface,
3. Proposes a non-uniform hardening rule for the subsequent loading surfaces with a hydrostatic pressure and Lode-angle dependent plasticity modulus, and
4. Utilizes a non-associated flow rule for a general formulation.

The important features of the inelastic behavior of concrete including: brittle failure in tension, ductile behavior in compression, hydrostatic pressure sensitivity and volumetric dilation under compressive loading, can all be represented by this refined constitutive model.

5.3. Strain-softening as a recent progress

Engineering materials such as concrete, rock and soil exhibit a strong strain-softening behavior in the post-peak stress range, showing a significant elastic–plastic coupling for the degradation of elastic modulus with increasing plastic deformation. Stress-space formulation of plasticity based on Drucker's stability postulate for these materials encounters difficulties in modeling the softening/elastic–plastic coupling behavior. Strain-space formulation is therefore necessary for further progress. As pointed out by Casey and Naghdi (1984) for some years, any arbitrary path in strain space can be specified independently by whether the material work hardens, is perfectly plastic, or strain softens. In this type of formulation the difference in material behavior can be easily described, and it permits a continuous description from one type of behavior to the other with ease. This is in contrast with the conventional stress space formulation for which the work-hardening and strain-softening behaviors must be treated differently. Although the representation of stress–strain behavior in either space can be translated into the other, the use of strain space formulation is more convenient for materials exhibiting the strain-softening behavior. On the other hand, the stress space formulation is often called for when we need a better physical understanding of the material behavior in terms of the applied stress and stress increments, that are normally used in our physical description of material behavior.

As an illustrative example, Han and Chen (1986) presented a consistent form of the constitutive relation for an elastic–plastic material with stiffness degradation in the range of work-hardening as well as strain-softening. Features of this approach include:

1. A relaxation surface is defined in strain space which serves as a criterion for further yielding and fracturing,
2. The dissipated energy due to plastic-fracturing is used as the parameter to record the material history and define both the evolution of the relaxation surface and the elastic degradation,
3. The weak stability postulate of Il'yushin's is used to obtain a relaxation rule, and
4. The consistency condition is used in establishing the constitutive relationships.

Details of the development can be found in the book by Chen and Han (1982).

5.4. *Plasticity on the miniscale as the current focus*

Strain-softening behavior may not be a material property and any formulation based on continuum mechanics may be misleading (Bažant and Belytschko, 1987). It has been shown that the extension of bond cracks is responsible for the nonlinear behavior of concrete and the growth of these cracks in the form of continuous cracks leads to failure in the case of uniaxial compression. This crack growth may be attributed to the sliding movement at the crack surfaces and to the sideway movement of aggregates. These mechanisms may result in irreversible (plastic) deformation and inelastic volume dilation (Yamaguchi and Chen, 1991). Further, the developments of these cracks could take place around many of the large aggregates. This is probably the reason why the overall stress–strain relationship in pre-peak stress regime provides an adequate representation of material properties in an average sense. However, bond cracks alone cannot cause failure since they are separated from one other. Failure occurs only when there are sufficient bond cracks interconnected with mortar cracks. The development of continuous crack patterns does not lead to immediate loss of load-carrying capacity because concrete at this stage behaves as a highly redundant structure. As successive load paths become inoperative through bond cracking, alternative load paths (either entirely through mortar, or partly through mortar and partly through aggregate) continue to be created and become available for carrying additional load. As the number of paths decreases, the intensity of stress and hence the magnitude of strain on the remaining paths increase at a faster rate than the external load. When the continuous crack pattern is developed extensively, the load-carrying paths are reduced considerably, resulting in a decrease of load-carrying capacity, and the descending branch of the stress–strain curve begins to form. This crack extension introduces various mechanisms including strain localization that govern the failure process of a specimen.

Microscopic observations can serve to reason out the stress–strain response and to capture its fundamental characteristics. Some fascinating applications of mechanics to concrete materials on the mini-scale have been reported in recent years. Details of this development can be found in the ASME state-of-the-art paper by Chen (1994c).

6. Design of steel structures with advanced analysis

When I first began to work in structural stability over 35 years ago, evaluation of the first-order response of a structural system was a significant problem. This includes linear elastic analysis and simple plastic analysis, and the progress made to the present state-of-the-art, which deals routinely with second-order inelastic analysis (commonly called *advanced analysis*) of complicated structural system having hundreds of thousands of degree of freedom, is miraculous. The modeling of all types of structural systems of high-rise buildings can now be handled quickly and efficiently on relatively inexpensive computers. The primary limitation is a sufficient understanding of the response of some secondary structural elements such as concrete floor slab, composite joints and walls that make up the system to develop simple but realistic models that can be incorporated into the analysis programs.

Research works are currently in full swing to develop nonlinear methods and software for practical use in the design office. The theory and approaches for advanced analysis of plane frames composed of members with compact sections, fully braced out-of-plane, have been well developed and verified by tests (Chen and Toma, 1994). Thus, it is feasible to model inelastic member and frame stability directly in a single analysis of planar frames. In fact the Australian Standard (AS4100, 1990) explicitly permits the checking of in-plane member and frame stability solely on the basis of advanced analysis.

Numerical tools for 3-D second-order inelastic analysis methods also have been proposed for analyzing and designing large-scale space frames. However, further work remains before the same can be said of modeling the more complex aspects of 3-D member behavior that involves inelastic lateral–torsional effects. At the present time, most practical approaches for stability design still require the separation of in-plane frame and member behavior from out-of-plane member stability checks. The members' slenderness ratios must be checked to ensure that lateral–torsional buckling does not occur. Furthermore, additional checks are required to ensure that inelastic rotation of plastic cross sections (or compact section) must have adequate rotational capacity to allow inelastic redistribution of forces between the frame members.

For advanced analysis to achieve its full potential as a tool for the design of steel frame structures, it must therefore have sufficient generality to cope with both effects, the local buckling of cross section and lateral–torsional buckling of member. At present, little attempt has been made to incorporate the effects of these two effects into a second-order inelastic analysis for practical frame design. Research needs to be accelerated on developing the advanced analysis tools for frame structures, where local buckling and inelastic lateral–torsional buckling become the limit states.

The analytical capability of tracing the performance of an in-plane frame structure into the nonlinear range including seismic loads is currently available. Advanced analysis combines the theory of stability with the theory of plasticity, and traces the gradual plastification of members with rigid or flexible joints in a steel frame. These developments can be found in the books by Chen and Toma (1994) and Chen and Sohal (1995). The power of this new development is the following: with the ability to predict the actual moment distribution at load levels which require members to sustain their plastic moment capacity, 'breaks' can be strategically located throughout the structure. These structural 'fuses' can be designed to fail themselves without risk of the building as a whole falling down, while leaving the majority of the connections in satisfactory condition. This would not only limit the amount of post-quake repair necessary, but also would indicate where the failed connections were and thus greatly reduce the expense of 'exploratory procedures'.

Direct second-order inelastic analysis for steel frame design consistent with the current steel design specifications can be found in the books by Chen and Kim (1997) and Chen et al. (1996), where the necessary software is also provided.

7. The finite block analysis for tension-weak materials

The finite element method is suitable for continuous materials that exhibit equally high tensile and compressive strength. Concrete and soils are strong in compression, weak in tension and become discontinuous materials when tensile cracks develop. The compatibility or continuity implied at the nodal points of the finite element approach requires significant modifications when applied to concrete materials.

The *Finite Block Method* deals with the equilibrium and kinematic of discontinuous block system separated through the existing or assumed crack surfaces in a structural analysis similar to a failure mechanism assumed in the classical upper limit analysis of perfect plasticity. Once a block system is assumed, a system of equilibrium equations for the block assemblage is derived through the minimization of the total potential energy. These equations are then solved by iteration in time increment under specific loading condition until the constraining requirements of no tension and no penetration between blocks are fulfilled. A complete kinematic theory of this type was developed by Shi in his Ph.D. thesis (Shi, 1988). Similar, but much earlier works on the subject area can be found in Cundall (1971) and Kawai (1977), among others.

The finite block method contains characteristic from both the upper and lower bound techniques of limit analysis of perfect plasticity. The method is similar to the upper bound technique in that a failure mode must be predetermined; block boundaries must be defined by the user; and the solution is based on energy equilibrium. The calculation of frictional dissipation requires that the normal force on the plane of sliding must be known before hand, thus, the finite block method provides an equilibrium stress field for all blocks at every stage of loading. Since Coulomb's friction law is enforced at the interface between blocks, the stress field so obtained is somewhat similar to the statically admissible stress field in the application of the lower bound theorem of limit analysis.

Thus, the finite block method for tension-weak materials may be considered as an extension of the computer-based limit analysis of perfect plasticity. The original version of finite block method is limited to constant-strain state in each block for simplification in the construction of equilibrium system. This may not be suitable for problems with large blocks or stress concentration. A higher order displacement function was therefore assumed and combined with the finite element technique for solving complex engineering problems. Practical applications of the finite block method include stability analysis and support designs for tunnels, slopes, retaining walls, dam abutments and foundations, etc. More detailed description and applications of this method may be found in the state-of-the-art paper by Chen and Huang (1994).

8. Concluding remarks

I believe the future direction of research and education in solid mechanics and structural engineering is in the area of '*modeling and simulation*'. In the current high-performance computing environment, the primary goal of this upcoming focused effort should be in the advancement of the state-of-the-art in a very complex scientific modeling and simulation and its validation for civil engineering applications. The major challenges are the integration of material science, structural engineering and computation and then demonstrating that the results are reliable. In the following, I shall list some of the challenging problems of structural engineering issues involving the interaction of mechanics, materials, and computing for the 21st century:

The Challenging Problems in Structural Engineering

- From Structural System Approach to Life-Cycle Analysis of Structures:
  - Construction: Sequence analysis
  - Service: Performance analysis
  - Degradation: Deterioration science
- From Finite Element Modeling for Continuous Media to Finite Block Analysis for Tension-Weak Materials:
  - Structures with changing geometry and topology
  - Materials with changing properties with time
- From Material Science Research to Structural Engineering Applications:
  - Micro-mechanics level
  - Continuum mechanics level
  - Structural engineering level

The Integrated Lifecycle Simulation of Civil Infrastructure

- Design
- Construction
- Service

- Deterioration
- Rehabilitation
- Demolition

The '*bottom line*' of my message on the future direction of structural engineering research and education is the following. We must de-emphasize the traditional narrow disciplinary approaches and increase integrative aspects of engineering involving *mechanics, materials, and computing*. We must emphasize that a true fulfillment of engineering research and education is '*a place in practice*'.

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